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Far Field Boundary Conditions for Compressible Flows*

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A family of boundary conditions which simulate outgoing radiation are derived. These boundary conditions are applied to the computation of steady state flows and are shown to significantly accelerate the convergence to steady state. Numerical results are presented. Extensions of this theory to problems in duct geometries are indicated.

I. INTRODUCTION

An important goal of computational fluid dynamics is the computation of steady state flows exterior to a body, such as an airfoil. This is frequently accomplished by integrating the time-dependent Navier–Stokes or Euler equations until a steady state is achieved. This raises two (related) computational problems:

(a) How to compute in an exterior region;

(b) How to accelerate convergence to the steady state.

The numerical treatment of exterior regions requires a method to convert the problem to one in a bounded region. One method is to map the exterior region into a finite region. Many equations, however, have oscillatory solutions near infinity. For these cases, the mappings can create substantial errors since waves in the vicinity of infinity cannot be resolved (see, e.g., Grosch and Orszag [1]). An alternative approach is to truncate the unbounded region at some finite, artifical surface. This creates a finite computational region at the expense of imposing boundary conditions at the artificial boundary.

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If the problem has wavelike solutions near infinity, then these boundary conditions must simulate the radiation of energy out of the computational domain and towards infinity. Incorrect specification of these radiation boundary conditions can cause spurious reflected waves to be generated at the artifical boundary. These waves represent energy propagating into the computational domain from infinity. Since they are not part of the desired solution, they can substantially degrade the accuracy of the computed solution. If the time-dependent equations are only an intermediate step towards computing a steady state, then a flow of energy into the computational domain can delay convergence. Conversely, the correct specification of radiation boundary conditions can accelerate convergence. Thus, the two questions raised in the first paragraph are related through the concept of radiation boundary conditions.

In order to minimize the effect of spurious reflections, the computational region can be enlarged so that the artificial boundaries are far from the region of interest. This will increase both the memory requirements and the running time of the computer program. It is preferable to specify boundary conditions that allow the computational region to be constricted as much as possible.

Our goal is therefore to construct radiation boundary conditions which have the following properties:

(a) To accurately similate the radiation of energy out of the computational domain.

(b) To accomplish (a) with an accuracy which improves as fast as feasible as the artificial surface is moved outward. Equivalently, the boundary conditions will be accurate when the computational domain is constricted.

(c) To accelerate convergence to the steady state (by minimizing spurious reflections).

The approach taken here is similar to that of [2–4]. We assume that in the far field the (possibly nonlinear) equations reduce to some simple form, e.g., a wave equation, Poisson's equation or the reduced wave equation. An asymptotic solution to the model equation is then constructed. This solution is usually based on a general functional form which specifies the behavior of the solution near infinity. In the problems considered here, the desired behavior is that the solution be composed of outgoing waves. We wish to stress that the expansion depends on the geometrical properties of the computational domain. For example, the asymptotic expansion will differ in ductlike geometries (infinite in only one dimension) from that in fully exterior regions (infinite in all dimensions). Thus, the expansion is based on global properties of the solution. The asymptotic expansions will usually be in terms of a reciprocal radius, i.e., in terms of distances, but can also be in terms of frequencies.

Once the functional form of the asymptotic expansion is known, we can derive differential relations that are exactly satisfied by any function having the given functional form. These differential relations, when used as boundary conditions, effectively match the solution to the asymptotic expansion valid near infinity. These radiation boundary conditions become increasingly accurate as they match the solution to more terms of the asymptotic expansion.

This procedure of matching the solution to an expansion valid near infinity requires some knowledge of the solution in a neighborhood of infinity. Gustaffson and Kreiss [5] have shown that problems in an exterior domain can in general be restricted to a bounded domain only when the dependent variables (and coefficients) approach constants at infinity. This suggests that radiation boundary conditions, to be applied at some artificial finite boundary, can work only by conveying information about the behavior of the solution near infinity.

In this paper, we shall concentrate on applications to fluid dynamics. In aerodynamics, viscous effects are important only in the vicinity of bodies. Hence, the far field behavior is governed by the Euler equations. A consequence of this is that solutions in the far field have a wavelike behavior. Hence, a steady state can be achieved only by allowing the radiation of energy outside the computational domain. Therefore, a more accurate simulation of the outward radiation of energy can accelerate the convergence to a steady state.

In the far field, the solution is relatively constant. Hence, to derive the boundary conditions, we linearize the Euler equations about this constant solution. The resultant system is equivalent to a convective wave equation. A family of radiation conditions for the standard wave equation was developed in [2]. This family was based on an expansion that was asymptotic in the distance from an arbitrary origin. A family of differental operators B_m was derived which annihilated the first m terms in the asymptotic expansion. An a priori estimate was obtained that showed that the error due to the use of an artificial surface coupled with the use of the boundary condition $B_m u = 0$ was $O(r^{-m})$.

The first member of this family was generalized to the convective wave equation in [3]. As stated above, this allows the construction of boundary conditions for the full time-dependent compressible Navier–Stokes equations. This condition will be discussed in more detail in this paper. In [4], these operators were generalized to elliptic equations such as the exterior Poisson and exterior Helmholtz equations.

Other approaches to the construction of outflow boundary conditions were developed by Rudy and Strikwerda [6, 7] and Engquist and Majda [8, 9]. Rudy and Strikwerda analyzed a one-dimensional model problem. A boundary condition was developed which accelerated the convergence to a steady state. This condition depended on a free parameter which was chosen, in the one-dimensional case, to maximize the convergence rate. In [6, 7], this boundary condition was applied to some two-dimensional problems where the free parameter was chosen by computational experimentation. This boundary condition was shown to substantially accelerate convergence to the steady state.

A different philosophy was adopted by Engquist and Majda [8, 9]. Their approach was to construct a pseudo-differential operator which exactly annihilated outgoing waves. This pseudo-differential operator was a global boundary operator. In order to derive local (i.e., differential) boundary operators, they expanded the pseudo-differential operators in the deviation of the wave direction from some preferred direction of propagation. In this manner, they constructed a family of local boundary conditions which absorbed waves in a progressively larger band around a given propagation direction.

These boundary conditions were tested by Kwak [10] on the time-dependent small disturbance equation. It was found that the first order condition significantly improved the standard condition ($\phi = 0$), where ϕ is the potential. The second-order condition was found to offer no significant advantages. (This was an accuracy study and not a steady state problem.) In this case, as in other cases with circular or spherical symmetry, the first-order Engquist and Majda condition and the first-order condition in [2] coincide.

We have so far concentrated on the use of radiation boundary conditions to accelerate convergence to steady state. There is also a great variety of problems which are inherently time dependent. These problems include the problem of acoustic radiation in a jet [11], problems in duct acoustics [12], and problems involving oscillations in the position of shocks [13]. In these types of problems, it is necessary to simulate the condition of outgoing radiation in order to obtain a correct numerical solution for the time scale of interest.

The theory to be presented here will be equally valid for these time-dependent (or time-harmonic) problems, provided there is some knowledge of the functional form of the solution near infinity. In Section II, we shall derive a family of boundary conditions designed to simulate outgoing radiation for the wave equation. This will lead to a radiation boundary condition applicable in the presence of a mean flow. This boundary condition shows great promise in accelerating flows to steady state. Numerical results illustrating this will be presented in Section III. In Section IV, we shall present extensions of this theory.

II. Development of Radiation Boundary Conditions

Consider the wave equation in three space dimensions

$$p_{tt} = \Delta p. \tag{1}$$

In a general inviscid flow, if p is the deviation of the pressure from the far field pressure p_∞, then p will satisfy equation (1) provided the free stream velocities are zero. (Throughout this paper we shall use the subscript ∞ to indicate free stream values.)

A spherical wave solution to Eq. (1) has the functional form

$$p = f(t - r)/r. \tag{2}$$

Here f is an arbitrary function and r is the distance from some fixed origin from which the spherical wave emanates. A boundary condition designed to simulate outgoing radiation should be exact at least for waves of form (2). Suppose the artificial boundary is the sphere $r = r_1$. A boundary condition which is exactly satisfied by all waves of form (2) is

$$B_1 p = 0, \tag{3}$$

where the operator B_1 is given by

$$B_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}. \tag{4}$$

Condition (3) can be regarded as matching the solution p to functional form (2).

In general, the waves impinging on the artificial surface are not exactly spherical. As an example, dipoles and quadrupoles often arise in aeroacoustics. In general, an outwardly radiating solution to Eq. (1) will have the asymptotic expansion

$$p \simeq \sum_{j=1}^{\infty} \frac{f_j(t-r, \theta, \phi)}{r^j}. \tag{5}$$

Here θ and ϕ are the angular variables associated with a spherical coorinate system centered at $r = 0$, while the functions f are arbitrary functions. (In principle, f_j for $j > 1$ can be determined from the radiation pattern f_1; however, for numerical purposes they should be treated as arbitrary functions.) The argument $t - r$ determines that the wave is outgoing, while the θ, ϕ dependence allows for a skewing from spherically symmetric waves.

Series (5) has been studied by many authors (see, e.g., Friedlander [14]). We are not concerned here with conditions for convergence, but merely require that this series represent the behavior of the solutions in a neighborhood of infinity.

Applying the operator B_1 to representation (5), we see that

$$B_1 p|_{r=r_1} = O(1/r_1^3). \tag{6}$$

It therefore follows that the radiation boundary condition

$$B_1 p = 0 \tag{7}$$

will be increasingly accurate as the position of the artificial boundary, e.g., the sphere $r = r_1$, approaches infinity. Condition (7) is exact only for the first term in expansion (5). It can be regarded as matching the solutions to the first term in expansion (5) with the error in (6) depending on the amount of skewing expressed in the next order term f_2.

Based on this motivation, a natural procedure to improve condition (7) is to derive boundary conditions which match the solution to the first two terms in (5). Such a condition is

$$B_2 p|_{r=r_1} = 0, \tag{8}$$

where the operator B_2 is given by

$$B_2 = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{3}{r}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\right) = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{2\partial^2}{\partial t\, \partial r} + \frac{4}{r}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\right) - \frac{2}{r^2}. \quad (9)$$

It can easily be verified that (8) is exactly satisfied by the first two terms in the expansion (5). Applying the operator B_2 to (5) we obtain

$$B_2 p|_{r=r_1} = \mathrm{O}\left(\frac{1}{r_1^5}\right). \quad (10)$$

Thus, boundary condition (8) is more accurate in the vicinity of infinity bcause it matches the solution to the first two terms in (5). Consequently, boundary condition (8) can be expected to be more accurate at near distances. Alternatively, if the solution is, for example, a pulse, (8) would be expected to accelerate convergence to the steady state value because the reflected waves generated at the boundary would be smaller.

The procedure which led to operator (9) can obviously be extended to boundary operators which annihilate any number of terms in (5). The well-posedness of the resulting boundary conditions together with rigorous error bounds are discussed in [2]. Some numerical results for acoustic radiation problems will be presented in the next section.

In order to derive boundary conditions for more general fluid dynamics problems, we next consider the effect of a constant free stream flow. For simplicity, we consider the two-dimensional Cartesian case. Let x and y denote the coordinate directions and u and v the corresponding velocity components. In addition, we let p be the pressure and ρ the density. We assume that in the far field the resulting steady state is given by

$$u = u_\infty, \qquad v = 0, \qquad p = p_\infty, \qquad \rho = \rho_\infty. \quad (11)$$

(This can always be arranged by a rotation of the coordinate system.)

In the far field, away from bodies and boundary layers, viscosity and entropy changes can be neglected. If we therefore introduce the deviations from steady state

$$\hat{u} = u - u_\infty, \qquad \hat{v} = v - v_\infty, \qquad \hat{p} = p - p_\infty, \qquad \hat{\rho} = \rho - \rho_\infty, \quad (12)$$

and assume that quadratic terms in these variables can be neglected, we obtain the linearized Euler equations

$$\begin{gathered} \hat{u}_t + u_\infty \hat{u}_x + \hat{p}_x/\rho_\infty = 0, \qquad \hat{v}_t + u_\infty \hat{v}_x + \hat{p}_y/\rho_\infty = 0, \\ \hat{p}_t + u_\infty \hat{p}_x + \rho_\infty c_\infty^2(\hat{u}_x + \hat{v}_y) = 0. \end{gathered} \quad (13)$$

(Here c_∞ is the far field sound speed.)

By straightforward manipulation, (13) can be reduced to a convective wave equation for $\hat{p}$,

$$\hat{p}_{tt} + 2u_\infty \hat{p}_{xt} - (c_\infty^2 - u_\infty^2)\hat{p}_{xx} - c_\infty^2 \hat{p}_{yy} = 0. \tag{14}$$

Assuming that the steady state is subsonic in the far field, i.e., $u_\infty < c_\infty$, we can derive radiation boundary conditions for (14) in the same manner as for the standard wave equation (1). At a subsonic boundary, one boundary condition must be imposed, either for (14) or system (13) (see [15]). Choosing a radiation boundary condition can be expected to accelerate the convergence to a steady state.

At a supersonic outflow boundary, all of the characteristics go out of the domain. No extra boundary condition can be imposed in this case, instead the solution should be obtained by a numerical procedure such as extrapolation.

We next proceed to derive a radiation boundary condition based on the convective wave equation (14). The most convenient way to proceed is to introduce new coordinates which transform (14) into the standard wave equation. Let M_∞ be the free stream Mach number,

$$M_\infty = u_\infty / c_\infty. \tag{15}$$

($M_\infty < 1$ is the condition for a subsonic boundary.) Introducing new coordinates

$$\xi = (1 - M_\infty^2)^{-1/2} x, \qquad \tau = c_\infty (1 - M_\infty^2)^{1/2} t + M_\infty \xi, \tag{16}$$

Eq. (14) is transformed into

$$\hat{p}_{\tau\tau} = \hat{p}_{\xi\xi} + \hat{p}_{yy}. \tag{17}$$

Equation (17) is the two-dimensional version of (1) and will therefore have outgoing circular waves, which in the two-dimensional case are asymptotic solutions. Introducing polar coordinates

$$d^2 = \xi^2 + y^2; \qquad \tan\theta = y/\xi, \tag{18}$$

then for large values of t and d, p will have the asymptotic form

$$\hat{p} \simeq f(\tau - d, \theta)/d^{1/2}. \tag{19}$$

(This was shown, for example, by Lax and Phillips in [16].)

In the same way as boundary operator (4) was derived, it can be shown that the boundary condition

$$\hat{p}_\tau + \hat{p}_d + \hat{p}/2d = 0 \tag{20}$$

is exact for all functions having functional form (19). We next express (20) in the physical coordinates t, x, and y. For clarity, we shall write it in terms of the total pressure $p = \hat{p} + p_\infty$. We obtain the condition

$$(1/c_\infty \sqrt{1-M_\infty^2})[1-(x/d)(M_\infty/\sqrt{(1-M_\infty^2})]\, p_t + (x/d)\, p_x + (y/d)\, p_y + (p-p_\infty)/2d = 0, \tag{21}$$

where in physical coordinates

$$d^2 = x^2/(1-M_\infty^2) + y^2. \tag{22}$$

(The factor of 2 in the last term in (21) is due to the $d^{1/2}$ decay in Eq. (19) which is characteristic of two-dimensional problems.) We finally use the linearized Euler equations (13) to eliminate the spatial derivatives of p. The result is

$$\frac{1}{(c_\infty^2 - u_\infty^2)^{1/2}}\, p_t - \frac{\rho_\infty c_\infty^2}{c_\infty^2 - u_\infty^2} \frac{x}{d}\, [u_t - u_\infty v_y] - \rho_\infty \frac{y}{d}\, [v_t + u_\infty v_x] + \frac{p - p_\infty}{2d} = 0. \tag{23}$$

In the steady state, (23) does not strictly enforce $p = p_\infty$, i.e., p equals p_∞ only when the gradient of v vanishes. Based on the functional form (19), any steady state must satisfy $p - p_\infty = O(d^{-1/2})$, and it can then be seen that at the steady state, (23) enforces $p = p_\infty$ to within $O(d^{-1/2})$, which is the accuracy of the boundary condition. In practice, the gradients of v are negligible and in fact, u and v are often obtained from zeroth order extrapolation, which is a consistent approximation to $v_x = 0$ and hence $v_y = 0$ (from Eq. (13)). The numerical results presented in Section III were generally obtained by neglecting the spatial derivatives of v in Eq. (23).

$$(1/(c_\infty^2 - u_\infty^2)^{1/2})\, p_t - (\rho_\infty c_\infty^2/(c_\infty^2 - u_\infty^2))(x/d)\, u_t - \rho_\infty (y/d)\, v_t + (p - p_\infty)/2d = 0. \tag{24}$$

The coefficients in (23) require knowledge of ρ_∞, and c_∞, which are not generally known a priori at the boundaries. We have used the values at the preceeding time step and have found no difficulties from this.

Boundary condition (24) is an outflow boundary condition which simulates outgoing radiation. When coupled with some numerical procedure for the other variables (typically zeroth order extrapolation), it permits a boundary treatment which sustantially accelerates convergence to the steady state. Our numerical experiments indicate that this acceleration is relatively insensitive to the choice of origin. From the nature of its construction, it is also applicable to truly time-dependent problems where there is a continuing radiation of energy across the outflow boundary.

Another application of this theory is at characteristic boundaries. These boundaries are tangent to the free stream velocities and arise even in supersonic flow. For example, in flow past a flat plate, with free stream velocity u_∞, $v_\infty = 0$, the boundary $y = \text{const}$ (i.e., the top of a computational rectangle) is characteristic. At these boundaries, one frequently extrapolates all the dependent variables. This can cause oscillations which delay convergence to a steady state. In addition, these oscillations require that the top boundary be sufficiently far away so that the oscillations will not degrade the accuracy of the steady state (see, e.g., [7]).

We have applied condition (24) at characteristic bounaries with success. Since no condition (other than $v = 0$) should be imposed at a characteristic boundary, better results were obtained by replacing p_∞ by p at the preceding time step. An alternative is simply to impose the one-dimensional characteristic condition

$$p_t - \rho_\infty c_\infty v_t = 0, \tag{25}$$

which is valid even if the basic flow is supersonic. The use of Eq. (25) can also dramatically improve convergence. The numerical results to be presented in the next section show the improvements that can be obtained by correctly treating the characteristic boundaries.

III. Numerical Examples

To validate boundary condition (24), we have studied several test problems. The first set of probmems is designed to study the rate of convergence to a steady state.

In the first problem, a compressible vortex is superimposed on a uniform flow $u = u_0$, $v = 0$ in a rectangular domain $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The vortex is modelled by

$$\begin{aligned} d^2 &= R^2 + (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2, \qquad R = 0.15, \\ u &= (\Gamma/\pi)(y - \tfrac{1}{2})/d^2, \\ v &= -(\Gamma/\pi)(x - \tfrac{1}{2})d^2, \\ \Gamma &= \text{circulation} = 2\pi R (v_\theta)_{\max}. \end{aligned} \tag{26}$$

The Mach number is found by assuming that the temperature is a fixed constant while the pressure is calculated from the total pressure by assuming isentropic flow. Analytically, the vortex convects downstream and out of the computational domain. The steady state is then just the free stream flow. Hence, the number of iterations required for convergence is directly related to the ability of the downstream boundary condition to allow the vortex to pass out of the domain without reflections.

This problem (which was suggested by David Rudy of NASA Langley) is a model for both steady state problems and time-dependent compressible Navier–Stokes equations. MacCormack's method is used to numerically solve the equations. The free stream Mach number is 0.4 and the Reynolds number is approximately 2100.

In Table I the number of iterations required for convergence is shown for different choices of the origin. Boundary condition (24) is compared with that suggested by Rudy and Strikwerda (see [6]) and also the boundary condition $p = p_\infty$. The steady state is assumed to occur when the change in all the dependent variables is less than 10^{-7}.

TABLE I

Number of Iterations to Achieve a Steady State for a Model Problem Containing a Compressible Vortex

Boundary conditions	Origin	Number of iterations
(24)	(0, 0.5)	2251
(24)	(0.5, 0.5)	2878
(24)	(0, 1.0)	2503
(24)	(0, 0.)	2530
Ref. [6]	—	7860
$p = p_\infty$	—	>20,000

The results show that (24) provides a significant acceleration to the steady state and that the number of iterations is relatively insensitive to the choice of origin. Similar results are obtained when the disturbance is a region of high pressure instead of a vortex (see [3, 6]).

As a more realistic test case, we consider the devlopment of a subsonic boundary layer over a flat plate. This problem was used in [7] to compare a wide range of outflow boundary conditions. The computational domain is a rectangle with the bottom corresponding to the plate. The top of the computational rectangle is a characteristic boundary. A standard boundary treatment at this upper surface is to extrapolate all of the variables. We also consider the effect of constricting this upper boundary. In Table II, the number of iterations required for convergence is shown for both boundary treatments and for different positions of the upper boundary. Results are also presented for the radiation boundary condition of [7] with the free parameter α chosen to be 0.3. Distances are measured in boundary layer thicknesses with the x coordinate varying between 0.0 and 2.0 and the maximum upper boundary chosen as 1.0.

TABLE II

Results for Boundary Layer over a Flat Plate with Extrapolation on Upper Boundary

Position of upper boundary	Boundary condition	Origin	Iterations
1.0	Eq. (24)	(1, 0)	12,500
1.0	Eq. (24)	(0, 0)	14,000
1.0	Ref. [7]	—	12,800
1.0	$p = p_\infty$	—	>20,000
0.6	Eq. (24)	(1, 0)	13,950[a]
0.4	Eq. (24)	(1, 0)	>14,000[a]

[a] Inaccurate steady state

The results presented in Table II demonstrate that all of the radiation conditions, i.e., (24) and those in [7] provide a substantial improvement over the condition $p = p_\infty$. The effectiveness of (24) is still relatively insensitive to the choice of origin. Constricting the upper boundary and using extrapolation led to osillations which delayed the attainment of the steady state. The final steady state in this case also differed significantly from the steady state obtained by integrating the equations in the larger region.

The use of a larger computational domain substantially increases the cost of the computation. This is particularly true because, in many cases, only the solution near the surface (in this case the plate) is of interest. In order to test the possibility of constricting the upper boundary, we applied radiation condition (24) at the upper boundary. In this case, we replace p_∞ by p at the previous time step, as the condition $p = p_\infty$ in the steady state is not valid when the upper boundary is reduced. In Table III, the results of applying (24) at the upper boundary are shown.

The data show that a substantial acceleration of convergence can be obtained by using a radiation condition at a characteristic boundary. We note that in all cases, the steady states in Table III were virtually identical to the steady states obtained on the larger regions. Thus, a substantial savings in computer time can be achieved by a correct choice of the boundary condition at the characteristic boundary. We have found similar improvements by using a characteristic condition such as (25) which can be used for supersonic flow as well. The reason that the boundary condition of [7] is not accelerated by a radiation condition at the characteristic boundary is not presently clear.

As another example, we consider inviscid flow in a quasi one-dimensional nozzle with variable area $A(x)$. The equations are

$$(A\rho)_t + (A\rho u)_x = 0, \qquad (A\rho u)_t + [A(\rho u^2 + p)]_x = A_x p, \qquad (AE)_t + [Au(E + p]_x = 0, \tag{27}$$

where E is the total energy. At the (subsonic) inflow, both u and E are specified. We shall only consider the case of a subsonic outflow so that one boundary condition

TABLE III

Effect of the Radiation Condition at the Upper Boundary

Position of upper boundary	Outflow boundary condition	Top boundary condition	Iterations
1.0	Eq. (24)	Eq. (24)	8800
1.0	Ref. [7]	Eq. (24)	12850
0.6	Eq. (24)	Eq. (24)	8800
0.4	Eq. (24)	Eq. (24)	9400

should be specified. By properly adjusting this outflow pressure, both shocked and shock-free solutions can be obtained.

We consider the effect of different radiation conditions on convergence to a steady state for system (27). In this case, the condition $p = p_\infty$ is valid during the temporal evolution of the flow. The use of a radiation condition is therefore a physically inconsistent artifice to accelerate the convergence to a steady state.

Since system (27) is one-dimensional, traveling waves have no spatial decay, in contrast to (2) or (19). Therefore, the appropriate generalization of (24) is the characteristic condition

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u}{\partial t} = 0. \tag{28}$$

This condition was tested along with the generalized radiation condition (see [6])

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u}{\partial t} + \alpha(p - p_\infty) = 0. \tag{29}$$

The solution method used was a linearized implicit Euler method (see [7, 8]. Boundary conditions on the pressure are computed by using an equation for $\partial p/\partial t$ and using a linearization technique similar to the basic difference scheme. In Table IV, we present results for several different choices of α. Both the number of iterations required for convergence and the steady state L_2 error are presented.

The results show that the radiation condition at the outflow can substantially accelerate convergence. In this case, it is not necessary to have a boundary condition which enforces $p = p_\infty$ in the steady state (i.e., $\alpha = 0$). The steady state condition $p = p_\infty$ is a consequence of the initial conditions. Similar results can be obtained by using the MacCormack explicit scheme (see [3]).

As an additional example, we consider the use of higher order boundary condition (10) in a time-harmonic problem. In this case, we compute the acoustic field due to a quadrupole source in a medium at rest. Quadrupole sources are important in the

TABLE IV

Steady State Results for the One-Dimensional Nozzle Flow

Boundary condition	L_2 steady state error	Iterations
$\alpha = \infty$ (i.e., $p = p_\infty$)	11.4	153
$\alpha = 0.0$	8.11	52
$\alpha = 0.278$	9.33	118
$\alpha = 1.0$	8.40	182
$\alpha = 10.0$	11.16	158

TABLE V

Phase Change for Quadrupole Source

Boundary condition	Phase change
Eq. (3)	0.77 ± 0.08
Eq. (10)	1.64 ± 0.08

theory of jet noise, and an effective computation requires boundary conditions which will not allow spurious reflections. First-order condition (3) is inaccurate in this case because the leading term in expansion (5) becomes very small at 90°.

In Table V, the phase change between 0° and 90° is presented for the two boundary conditions (3) and (10). The analytic solution should exhibit a phase change of $\pi/2$ (i.e., from a cosine to a sine dependence as the angle changes from 0° to 90°. It can be seen that second-order boundary condition (10) gives an accurate phase change while (3) gives a completely wrong phase change. This result clearly indicates the crucial importance of radiation boundary conditions in simulating fluctuating flows such as those occurring in aeroacoustics.

IV. Extensions to Time-Independent Equations

In the previous sections, we have concentrated on developing boundary conditions for time-dependent equations. These conditions were used even in cases where the time dependence was only a mechanism for achieving the steady state solution. There are many applications, however, when one solves the time-independent equations directly. Generally in those cases a time-harmonic solution is sought. Some examples of such aerodynamic problems are the small disturbance equation about a fluctuating airfoil [19], acoustic propagation in a duct [20], and the scattering of accoustic waves by a body such as an airplane fuselage [21]. In this section, we shall consider the problem of deriving radiation boundary conditions for such problems.

As a model equation, we consider the Helmholtz equation

$$\Delta\phi + k^2\phi = 0. \tag{30}$$

(In applications, ϕ can be a velocity potential or a fluctuating quantity such as the pressure.) When (30) is considered exterior to a body, it describes the scattering of waves by the body. When (30) is integrated in a duct, it describes the propagation of acoustic (or electromagnetic) waves in the duct. In each case, (30) must be supplemented by appropriate boundary conditions on the physical surfaces and a radiation condition enforcing outgoing radiation at infinity.

The appropriate radiation boundary condition is different in fully exterior geometry than in duct geometry. This occurs since, in the duct geometry, there can exist a finite number of propagating modes, each with a different wavenumber. In fully exterior regions, on the other hand, radiation propagates radially (with some angular skewing), and there is essentially only one wavenumber.

In the fully exterior region, the appropriate boundary condition at infinity is the Sommerfeld radiation condition

$$\phi_r - ik\phi + \phi/r = O(1/r) \qquad (r \to \infty). \tag{31}$$

This condition can be derived by formally differentiating the expansion

$$\phi = e^{-ikr} \sum_{j=1}^{\infty} \frac{f_j(\theta, \phi)}{r^j}, \tag{32}$$

which is the direct analog of (5) for the case of a harmonic time dependence. Higher-order boundary conditions can be obtained by matching the solution to more terms in the expansion of Eq. (32). For example, a second-order boundary operator, analogous to (9) is

$$\begin{aligned} B_2 &= \left(-ik + \frac{\partial}{\partial r} + \frac{3}{r}\right)\left(-ik + \frac{\partial}{\partial r} + \frac{1}{r}\right) \\ &= -k^2\phi + \frac{\partial^2\phi}{\partial r^2} - \left(2ik - \frac{4}{r}\right)\frac{\partial\phi}{\partial r} - \left(\frac{4ik}{r} - \frac{2}{r^2}\right)\phi. \end{aligned} \tag{33}$$

The properties of these boundary conditions are discussed in detail in [4].

A different situation occurs in the duct Helmholtz equation. For simplicity, we consider the Helmholtz equation in the rectangular region $0 \leqslant x < \infty$, $0 \leqslant y \leqslant \pi$. The equations are

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \tag{34a}$$

$$\partial\phi/\partial y = 0, \qquad \text{at} \quad y = 0, \pi, \tag{34b}$$

$$\phi|_{x=0} = f. \tag{34c}$$

It is easy to see that the general solution is

$$\phi = \sum_{j=0}^{\infty} A_j \cos(jy)[e^{i\sigma_j x} a_j + b_j e^{-i\sigma_j x}], \qquad \text{where} \quad \sigma_j = \sqrt{k^2 - j^2}. \tag{35}$$

We see that for $j \leqslant k$, the outgoing solution is obtained by choosing

$$\phi \sim e^{+i\sigma_j x}. \tag{36}$$

The modes with $j \geqslant k$ are evanescent (i.e., exponentially decaying) and the correct solution is obtained by requiring exponential decay as $y \to \infty$, i.e.,

$$\phi \sim e^{-\sqrt{(j^2 - k^2)}x}. \tag{37}$$

Thus, the wave numbers of the solution at infinity vary with the mode.

A boundary operator which is exact for the propagating modes is

$$B\phi = \prod_{j=1}^{[k]} \left(\frac{\partial}{\partial x} - i\sigma_j\right) \phi. \tag{38}$$

It can be shown that imposition of the boundary condition

$$B\phi|_{x=x_1} = 0, \tag{39}$$

accurately simulates the outgoing solution with an error that decreases exponentially as $x \to \infty$.

If problem (34) is fomulated in cylindrical geometry, it will describe waves propagating in a hard-walled duct. Introducing cylindrical coordinates r and z, where r is normal to the duct centerline and z is the distance along the axis, the problem becomes (the duct diameter is scaled to unity),

$$\phi_{zz} + (1/r)(r\phi_r) + k^2\phi = 0, \tag{40a}$$

$$\phi_r = 0, \quad \text{at} \quad r = 0 \quad \text{and} \quad r = \tfrac{1}{2}, \tag{40b}$$

$$\phi|_{z=-L} = f. \tag{40c}$$

The solution is

$$\phi = \sum A_j J_0(\sigma_j r)\, e^{+i\sigma_j z}, \tag{41}$$

where J_0 is the zeroth-order Bessel function and λ_j is twice the jth zero of J_0'; also

$$\sigma_j = \sqrt{k^2 - \lambda^2}. \tag{42}$$

It can be seen that boundary condition (39) is equally valid in this case. There is no need for the ducts to have straight walls except in the vicinity of infinity.

Similar boundary conditions can be developed for computing spin modes, ducts with flow, or for computing in ducts with liners, where condition (40c) is replaced by an impedence boundary condition. The second-order boundary condition can be implemented in finite element codes in a manner similar to that described in [4].

This theory has also been applied to underwater acoustics. A typical problem here is

$$\phi_{zz} + [(r\phi_r)_r/r] + k^2\phi = 0 \tag{43a}$$

$$\phi = 0, \quad \text{on} \quad z = 0, \tag{43b}$$

$$\phi_z = 0, \quad \text{on} \quad z = \pi, \tag{43c}$$

$$\phi = f, \quad \text{on} \quad r = r_c. \tag{43d}$$

This problem differs from (40) in that the computational domain is infinite in the r direction. The outwardly radiating solution is

$$\phi = \sum_{j=1}^{\infty} A_j \sin(\lambda_j z)\, H_0^+(\sigma_j r), \qquad \lambda_j = j - \tfrac{1}{2}, \quad \sigma_j = \sqrt{k^2 - \lambda_j^2}. \tag{44}$$

(See, e.g., Fix and Marin, [22].) Here H_0^+ is the Hankel function of zeroth order and of the first kind. By using the asymptotic expansion

$$H_0^+(z) \simeq \sqrt{2/\pi}\, e^{i(Z - \pi/4)}[(1/\sqrt{z}) + O(z^{-3/2})], \tag{45}$$

and the definition of σ_j, we see that the solution is composed of a finite number of propagating modes ($\lambda_j < k$) and an infinite number of evanescent modes ($\lambda_j > k$).

Fix and Marin have developed an exact global radiation condition for problem (43). This condition can be applied at any artificial boundary $r = r_1$. Local approximations can be developed using the ideas presented here. If there are m propagating modes, we can use the boundary condition

$$B_m(r_1)\phi = 0, \tag{46}$$

where B_m is the unique, mth-order differential operator which has as its fundamental set of solutions $\{H_0^+(\sigma_j r);\ j = 1,..., m\}$. Such an operator can easily be constructed using the theory of ordinary differential equations.

Simpler boundary conditions can be constructed by accepting an error of $r^{-3/2}$ and using the leading order term in expansion (45). In this case, we take as the operator B_m, the operator which has the fundamental set of solutions $\{e^{i\sigma_j r}/\sqrt{r},\ j = 1,..., m\}$. For the case $m = 2$, we get

$$B_2(r) = \left(\frac{\partial}{\partial r} - i\sigma_2 + \frac{1}{2r}\right)\left(\frac{\partial}{\partial r} - i\sigma_1 + \frac{1}{2r}\right). \tag{47}$$

This second-order boundary condition has been applied to various problems.

The theory presented here is also valid in the case of a variable sound speed or of varying topography of the ocean bottom. The second-order boundary condition can be easily implemented in variational principles as described in [4]. An efficient implementation of the higher-order boundary conditions must still be developed. Similar boundary conditions have been proposed by Kriegsmann [23] for waveguide problems in a Cartesian coordinate system.

V. Conclusion

We have derived boundary conditions which can be used on the artificial boundaries that arise when an unbounded region is truncated for computational purposes. These boundary conditions are based on matching the solution to a known functional form valid near infinity.

These boundary conditions have been applied to the nonlinear compressible Navier–Stokes and Euler equations. They have been shown to yield a substantial acceleration of convergence to the steady state. Over or under specification can lead to oscillations which degrade the accuracy of the steady state. The radiation boundary conditions can be used at both subsonic outflow boundaries and at characteristic boundaries where the normal velocity is zero.

The radiation boundary conditions can also be developed directly for the steady state equations. For subsonic flows, the equations are elliptic. Thus we are led to the development of Sommerfeld-type radiation conditions for elliptic equations. As before, these allow for the constriction of the domain of integration without loss of accuracy. It is also seen that the appropriate boundary conditions depend on the geometry of the region. Hence, even though the boundary conditions are local they depend on global properties of the solution. This occurs since the boundary conditions are developed based on asymptotic solutions valid in the vicinity of infinity. These asymptotic expansions depend on global properties of the solution. In particular, the geometry of the region in the far field strongly affects the proper choice of boundary conditions. This is true for both the steady state and time-dependent problems.

The boundary conditions developed are all local, i.e., differential boundary conditions. It is also possible to incorporate the asymptotic expansion directly in a finite difference scheme. This will lead to a relationship between the outermost grid points. This relationship is just the finite difference analog of the differential boundary conditions.

Numerical results have been presented here and in [2–4, 11, 24] which verify the usefulness of the proposed boundary conditions.

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